Strong WQO phase transitions[1]

Preliminary version

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Extended abstract

1 Summary

We investigate phase transitions for well-quasi-ordering (also referred to as *well-partial-ordering*, abbr.: *wpo*) results with respect to nested finite sequences and nested finite trees under the homeomorphic embedding with symmetrical gap condition. We consider three *wpo* spaces:

1. nested finite sequences,
2. finite trees labeled by nested finite sequences,
3. nested finite trees.

With every *wpo* $\mathcal{W}$ in question we correlate a natural PA-extension T (below $\Pi^1_1\mathsf{TR}_0$), that proves the corresponding 2-order sentence $\mathsf{WPO}\,(\mathcal{W})$. Furthermore, we consider the appropriate parametrized 1-order *slow well-partial-ordering* sentence $\mathsf{SWP}\,(\mathcal{W},\cdots,x)$ with x ranging over computable reals and actually compute a real number λ and prove that the following hold:

1. if $x < \lambda$ then $\mathsf{SWP}\,(\mathcal{W},\cdots,x)$ is provable in PA,
2. if $x > \lambda$ then $\mathsf{SWP}\,(\mathcal{W},\cdots,x)$ is not provable in T.

Such (uniquely determined) λ is called *phase transition* for $\mathsf{SWP}\,(\mathcal{W},\cdots,x)$. In limit cases we replace computable reals r by computable functions $f : \mathbb{N} \to \mathbb{R}$ and prove analogous theorems. These results strengthen familiar Kruskal-Friedman-Gordeev-Kriz wpo theorems and Weiermann's phase transitions concerning Kruskal-Friedman-Schütte-Simpson cases.

2 Preliminaries

2.1 Partial and linear well orderings

By $\trianglelefteq$ and $\leq$ we denote partial and linear countable well orderings (abbr.: *wpo* and *wo*), respectively. A *wo* $\mathcal{O} = (W, \leq)$ is called a *linearization* of a *wpo*

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$\mathcal{W} = (W, \trianglelefteq)$ iff $(\forall x, y \in W)\,(x \trianglelefteq y \rightarrow x \leq y)$. A *wpo* $\mathcal{W} = (W, \trianglelefteq)$ is called *enumerated* iff it is supplied with a bijection, also called *enumeration*, $\nu : \mathbb{N} \rightarrow W$. For any enumerated *wpo* $\mathcal{W} = (W, \nu, \trianglelefteq)$ we fix its lexicographical linearization $\mathcal{W}_\nu = (W, \leq_\nu)$ that is defined as follows

$$\begin{aligned} W \times W \quad \ni \quad & x \leq_\nu y :\Leftrightarrow (\forall i \in \mathbb{N})\,(\nu(i) \trianglelefteq x \longleftrightarrow \nu(i) \trianglelefteq y) \vee (\exists i \in \mathbb{N}) \\ & (\nu(i) \ntrianglelefteq x \wedge \nu(i) \trianglelefteq y \wedge (\forall j < i)\,(\nu(j) \trianglelefteq x \longleftrightarrow \nu(j) \trianglelefteq y)) \end{aligned}$$

2.2 Partial orderings on labeled sequences and trees

Definition 1 *A labeled finite tree T is* embeddable *with symmetrical gap condition into a labeled finite tree T' (abbr.: $T \trianglelefteq T'$) iff there is a homeomorphic embedding $h : T \rightarrow T'$ such that $(\forall x \in T)\,(x \leq h(x))$ and $\min\limits_{\leq}(x, x') \leq y$ holds for any two neighbors x, x' in any path $P \subset T$ and any $y \in h(P) \subset T'$. Here $\leq$ stands for the underlying wo on the set of labels occurring in T, T'. Note that $\trianglelefteq$ is a wpo. 1-D trees, i.e. trees without branchings, are called labeled sequences. Embedding of labeled finite sequences is an obvious 1-D specialization of general definition for trees. All finite structures in question are supplied with natural enumerations (ν) and norm functions ($\#$).*

Remark 2 *Without loss of generality we can just as well replace vertex-labeled trees by the corresponding edge-labeled trees.*

2.3 Nested sequences and trees

By $\textsc{Seq}^d$ we denote the set of nested finite sequences, i.e. finite sequences, finite sequences labeled by finite sequences, etc., where d stands for the depth of nesting in question. By $\trianglelefteq_d^1$ and $\leq_d^1$ we denote the corresponding embedding relation and its linearization, respectively, which are defined by simultaneous recursion on d. Thus $\textsc{Seq}^1 \cong \mathbb{N}$ and $\textsc{Seq}^2$ corresponds to the set of finite sequences with labels from $\mathbb{N}$.

By $\textsc{Ts}^d$, $\trianglelefteq_d^2$ and $\leq_d^2$ we denote the set of finite trees with labels from $\textsc{Seq}^d$, the corresponding embedding relation and its linearization, respectively.

By $\textsc{Tt}^d$ we denote the set of nested finite trees, i.e. finite trees, finite trees labeled by finite trees, etc., where d stands for the depth of nesting in question; the corresponding *wpo* $\trianglelefteq_d^3$ and *wo* $\leq_d^3$ are defined by simultaneous recursion on d. In particular $\textsc{Tt}^0 = \textsc{Ts}^0$ is the set of plain (unlabeled) finite trees.

3 Basic results

For any *wpo* $\mathcal{W} = (W, \trianglelefteq)$ let $\mathsf{WPO}(\mathcal{W})$ be an abbreviation of "$\mathcal{W}$ *is a wpo*" in the form $(\forall f : \mathbb{N} \rightarrow W)\,(\exists i < j \in \mathbb{N})\,(f(i) \trianglelefteq f(j))$. For any *wpo* $\mathcal{W} = (W, \trianglelefteq)$, $0 < d \in \mathbb{N}$ and $0 \leq r \in \mathbb{Q}$, let $\mathsf{SWP}(W, \trianglelefteq, d, r)$ be the corresponding first order refinement of $\mathsf{WPO}(\mathcal{W})$

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\begin{aligned}
&(\forall K \in \mathbb{N})\,(\exists M \in \mathbb{N})\,(\forall x_0, \cdots, x_M \in W)\\
&((\forall i \leq M)\,(\#(x_i) \leq K + r \cdot \lceil \log_d (i+1) \rceil) \rightarrow (\exists i < j \leq M)\,(x_i \trianglelefteq x_j))
\end{aligned}
$$

Theorem 3 *1. If $r < 1 \leq d$ then* $\mathsf{PA} \vdash \mathsf{SWP}\left(\textsc{Seq}^2, \trianglelefteq^1_d, d, r\right)$.

2. If $r > 1$ then $\mathsf{PA} \nvdash \mathsf{SWP}\left(\textsc{Seq}^2, \trianglelefteq^1_2, 2, r\right)$ *and* $\Delta^1_1\mathsf{CA} \nvdash \mathsf{SWP}\left(\textsc{Seq}^3, \trianglelefteq^1_3, 3, r\right)$.

Theorem 4 *For any computable $f : \mathbb{N} \rightarrow \mathbb{Q}$ and $I(x) := x$, the following holds.*

1. *If $f \prec^\infty I$ then* $\forall k\,(\exists d > k)\,\mathsf{PA} \vdash \mathsf{SWP}\left(\textsc{Seq}^d, \trianglelefteq^1_d, d, f(d)\right)$.
2. *If $I \prec^\infty f$ then:*
   - *(a)* $(\exists k)\,\mathsf{ATR}_0 \nvdash (\forall d > k)\,\mathsf{SWP}\left(\textsc{Seq}^d, \trianglelefteq^1_d, d, f(d)\right)$,
   - *(b)* $\mathsf{ATR} \vdash (\forall d > 0)\,\mathsf{SWP}\left(\textsc{Seq}^d, \trianglelefteq^1_d, d, f(d)\right)$.

Let $\rho(d) := \frac{1}{8}\left(-d + \sqrt{d^2 + 16}\right)$ and $\ell(d) := -\log_{\rho(d)}(2)$. Note that $\ell(1) \approx .7369095552$, $\ell(2) \approx .5902344834$, while $\ell(d) \rightarrow 0$ as $d \rightarrow \infty$. For brevity we also identify $\ell(1)$ with constant function $\ell(1) : \mathbb{N} \ni x \mapsto \ell(1) \in \mathbb{R}$. By $\mathsf{ID}^{(d)}_{<\omega}$ we denote a formal system that is defined by recursion $\mathsf{ID}^{(1)}_{<\omega} := \mathsf{ID}_{<\omega}$, $\mathsf{ID}^{(d+1)}_{<\omega} := \mathsf{ID}_{<o\left(\mathsf{ID}^{(d)}_{<\omega}\right)}$. Note that $\mathsf{ID}_{<\omega} = \Pi^1_1\mathsf{CA}_0$ and $\mathsf{ID}^{(d)}_{<\omega} \rightarrow \Pi^1_1\mathsf{TR}_0$ as $d \rightarrow \infty$.

Theorem 5 *For any $d > 1$ the following holds, where $\varphi_0(0) := 1$, $\varphi_1(0) := \omega$, $\varphi_2(0) := \varepsilon_0$, $\varphi_{d+1}(0) := \varphi(\varphi_d(0), 0)$; thus $\varphi_d(0) \rightarrow \Gamma_0$ as $d \rightarrow \infty$.*

1. *If $r \leq \ell(d)$ then* $\mathsf{PA} \vdash \mathsf{SWP}\left(\textsc{Ts}^d, \trianglelefteq^2_d, 2, r\right)$.
2. *If $r > \ell(d)$ then* $\mathsf{ID}_{<\varphi_d(0)} \nvdash \mathsf{SWP}\left(\textsc{Ts}^d, \trianglelefteq^2_d, 2, r\right)$.

Theorem 6 *For any $d \geq 0$ the following holds.*

1. *If $r \leq \ell(1)$ then* $\mathsf{PA} \vdash \mathsf{SWP}\left(\textsc{Tt}^d, \trianglelefteq^3_d, 2, r\right)$.
2. *If $r > \ell(1)$ then* $\mathsf{ID}^{(d+1)}_{<\omega} \nvdash \mathsf{SWP}\left(\textsc{Tt}^d, \trianglelefteq^2_d, 2, r\right)$.

Theorem 7 *For any computable $f : \mathbb{N} \rightarrow \mathbb{R}$ the following holds.*

1. *If $f \prec^\infty \ell(1)$ then* $\forall k\,(\exists d > k)\,\mathsf{PA} \vdash \mathsf{SWP}\left(\textsc{Tt}^d, \trianglelefteq^3_d, 2, f(d)\right)$.
2. *If $\ell(1) \prec^\infty f$ then:*
   - *(a)* $(\exists k)\,\Pi^1_1\mathsf{TR}_0 \nvdash (\forall d > k)\,\mathsf{SWP}\left(\textsc{Tt}^d, \trianglelefteq^3_d, 2, f(d)\right)$,
   - *(b)* $\Pi^1_1\mathsf{TR} \vdash (\forall d > 0)\,\mathsf{SWP}\left(\textsc{Tt}^d, \trianglelefteq^3_d, 2, f(d)\right)$.